Geometric realisation of Connes spectral triples for algebras with central bases

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Outline

NCRG statement for quantum gravity

1. We will not assume that the spacetime is continuum at the plank scale.
2. Instead propose that it is more effectively described by a noncommutative coordinate algebra.
3. And in the limit classical RG is recovered.

[1] E.J. Beggs and S. Majid, *Quantum Riemannian Geometry*, Grundlehren der mathematischen Wissenschaften, Vol. 355, Springer (2020) 809pp

Basic NCRG formalism

We work with A a unital algebra, typically a $*$-algebra over $\mathbb{C}$, in the role of 'coordinate algebra'.

1. **Differentials** are formally introduced as a bimodule Ω^1 of 1-forms equipped with a map $\mathrm{d} : A \to \Omega^1$ obeying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$.
2. Assume this **extends** to an exterior algebra (Ω, d) with $\mathrm{d}^2 = 0$ and d obeying the graded-Leibniz rule and $\wedge(g) = 0$.
3. A **quantum metric** is $g \in \Omega^1 \otimes_A \Omega^1$ and a bimodule map inverse $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$.
4. A **bimodule connection** on Ω^1 is $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ obeying:
    1. $\nabla(a.\omega) = a.\nabla\omega + \mathrm{d}a \otimes \omega$,
    2. $\nabla(\omega.a) = (\nabla\omega).a + \sigma(\omega \otimes \mathrm{d}a)$

    with σ a unique 'generalised braiding' bimodule map $\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$.

Basic NCRG formalism

Finally we define the quantum Levi-Civita connection (QLC) if

1. Is torsion free: $T_\nabla := \wedge\nabla - \mathrm{d}$ vanishes.
2. Metric compatible: $\nabla g = (\nabla \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \nabla))g$.

And the weak version (WQLC) if:

1. Is torsion free.
2. Is cotorsion free: $(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \wedge \nabla)g = 0$.

General axiomatic motivation

In recent years, the **quantum Riemannian geometry** was extended to a systematic theory including the QLC and further structure as:

1. 'spinor' bimodule $\mathcal{S}$ equipped with a bimodule connection ∇_S
2. A 'Clifford action' $\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}$.
3. Leading to a quantum-geometric Dirac operator $D = \triangleright \circ \nabla_S$.
4. And a inner product used to complete $\mathcal{S}$ to a Hilbert Space.

Motivation for central bases

We say a basis e^i is central if is a grassmann algebra: $e^i e^j + e^j e^i = 0$.

We will think in central bases if A has trivial centre, Ω^1 has a central basis $\{s^i\}$ and $\mathcal{S}$ has a central basis $\{e^\alpha\}$.

The central bases assumtion resumes into the set of 1-forms are self adjoint. The metric translates to the matrix g_{ij} of metric coefficient in the basis being hermitian. If σ is the flip map then the $*$-preserving condition on ∇ translates to the Christoffel symbols in the basis being real.

Spectral triples: algebraic features

We have important elements:

1. **Clifford action** $\triangleright : \Omega^1 \otimes_A \mathcal{S} \to \mathcal{S}$, $s^i \triangleright e^\alpha = C^{i\alpha}{}_\beta e^\beta$, $\quad C^{i\alpha}{}_\beta \in \mathbb{C}$.
2. The **antilinear map** $\mathcal{J}(ae^\alpha) = a^* J^\alpha{}_\beta e^\beta$, with $\overline{J}J = \epsilon \mathrm{id}, \quad \epsilon = \pm 1, \quad J^\alpha{}_\beta \in \mathbb{C}$
3. The bimodule connection and braiding: $\nabla_{\mathcal{S}} e^\alpha = S^\alpha{}_{i\beta} s^i \otimes e^\beta, \quad \sigma_S(e^\alpha \otimes s^j) = \sigma_S^{\alpha j}{}_{i\beta} s^i \otimes e^\beta, \quad \sigma_S^{\alpha j}{}_{i\beta} \in \mathbb{C}$.
4. The compatibility of the connection with $\mathcal{J}$ and σ_S: $\overline{S_j} J \sigma_{S\,i}^{j} = J S_i \qquad \overline{C^i} J = \epsilon' J \sigma_{S\,j}^{i} C^j$

For an **even** case we need an extra **bimodule map** γ:

$$\gamma^2 = \mathrm{id},$$
$$\{C^i, \gamma\} = 0,$$
$$\overline{\gamma} J = \epsilon'' J\gamma, \quad \epsilon'' = \pm 1,$$
$$[S_i, \gamma] = 0.$$

Spectral Triples: Geometric conditions

* The covariance $\nabla(\triangleright) = 0$ condition:
  $C^i S_j - \sigma^{ik}{}_{jl} S_k C^l = -\frac{1}{2}\Gamma^i{}_{jk} C^k$
* The compatibility with the Clifford action with Ω^2:
  $(s^i \wedge s^j)\triangleright e^\alpha := s^i \triangleright (s^j \triangleright e^\alpha) - g^{ij} e^\alpha$.

For the inner product we will assume a reference positive linear functional $\int : A \to \mathbb{C}$ and set

$$\langle \phi_\alpha e^\alpha, \psi_\beta e^\beta \rangle = \int \phi^*_\alpha \mu^{\alpha\beta} \psi_\beta$$

for some positive hermitian matrix μ as 'measure' and $\phi, \psi \in \mathcal{S}$.

Q:Difference with Connes?

A: Geometric conditions dont exist in Connes formalism and generalise the theory.

Small note

The Quantum geometric Dirac operator construction applied to an arbitrary unital $*$-algebra over $\mathbb{C}$ may or may not obey the axioms of a spectral triple. For instance

1. For the q-sphere, $C_q[S^2]$, we obtain a q-deformed Dirac Operator, because J is not fully an antilinear isometry
2. But for the fuzzy sphere, $C_\lambda[S^2] = U(su_2)/\langle x_i^2 = (1-\lambda^2)\;\; i \in \{1,2,3\}\rangle$, has unique modulo unitary equivalence natural spectral triple.

Some results and examples

Theorem

Up to a phase in the 2D spinor bundle case, J can be obtained with $r > 0$ and $z \in \mathbb{C}$ as either:

$$(1): \quad J = \begin{pmatrix} z & r \\ \frac{\epsilon - |z|^2}{r} & -\bar{z} \end{pmatrix}, \quad (2): \quad J = \begin{pmatrix} 1 & \frac{\epsilon - 1}{|z|^2} z \\ z & -\frac{z}{\bar{z}} \end{pmatrix}$$

or its transpose. The $\epsilon = -1$ case of (2) needs $z \neq 0$ and up to a phase is also an instance to type (1).

Some results and examples

Noncommutative Torus

the only geometrically realised Dirac operator for the standard Euclidean metric and ∇ a WQLC are

$$D(\psi_\alpha e^\alpha) = (\partial_i \psi_\alpha s^i) \triangleright e^\alpha + \psi_\alpha d_i s^i \triangleright e^\alpha = \sigma^{i\alpha}{}_\beta ((\partial_i + d_i)\psi_\alpha) e^\beta . \quad (1)$$

. With Hilbert space, the state $\int u^m v^n = \delta_{m,0}\delta_{n,0}$. Up to unitary equivalence, $(D\psi)_\beta = (\partial_i \psi_\alpha)\sigma^{i\alpha}{}_\beta$ is the only possibility for a geometrically realised spectral triple on the noncommutative torus for the Euclidean metric, a WQLC and the standard Hilbert space structure on $\mathcal{S}$.

Conclusions

Future path?

- One goal is to fully characterise central bases algebras by computing more examples, like q-deformed NC torus.
- Interpret the geometric realisation restrictions in particle physics constraints.
- Extend this to spinors and spectral triples without the Dirac Operator.

The end

Thank you for your attention!

Questions?